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Note for the SNU-APCTP winter school Feb.2-6, 2026

◦ Semi-classical approach to confinement inspired from generalized symmetry

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(* This note is prepared just to confirm if I can do the necessary computations/presentations on the blackboard. Thus, this is far from complete by definition...)

Outline :

1. Generalized symmetry. (p. 1–8)
   - 1-form symmetry & its gauging
   - Anomaly matching
2. Semiclassics for quantum mechanics (p. 9–13)
   - Particle on S^1 w/ θ-term
   - WKB (or dilute instanton gas) computation & energy spectrum
   - Symmetry & anomaly
3. Semiclassics for SU(N) Yang-Mills on $\mathbb{R}^2 \times T^2_{\text{twisted}}$ (p. 14 – 27)
   - YM on compactified space
   - Fractional instantons
   - Area law of Wilson loop, Multi-branch θ-vacua
4. Extension for QCD (p. 28 – 36)
   - $U(1)_{\text{Baryon}}$ symmetry & 't Hooft flux
   - QCD on $\mathbb{R}^2 \times (T^2)_{U(1)_B\text{-flux}} = \mathbb{Z}_N$-Schwinger model
   - Derivation of chiral Lagrangian & Fractionalized Kobayashi-Maskawa-'t Hooft determinant.
   - η'-vortex, Komargodski's pancake baryon scenario in large-N, & semiclassical derivation

Refs
- Tanizaki, Ünsal "Center vortex and confinement in Yang-Mills theory and QCD with anomaly-preserving compactification" 2201.06166
- Hayashi, Tanizaki "Semiclassics for the QCD vacuum structure through T^2-compactification with the baryon-'t Hooft flux" 2402.04320
- Hayashi, Tanizaki "Unifying Monopole and Center Vortex as the Semiclassical Confinement Mechanism" 2405.12402
- Hayashi, Tanizaki, Ünsal "Center-vortex semiclassics with non-minimal 't Hooft fluxes on $\mathbb{R}^2 \times T^2$ and center stabilization at large-N" 2505.07467

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§1. Generalized Symmetry

Conventionally,

Symmetry = invariance of Lagrangian $\mathcal{L}$ (up to total derivatives)

$\Downarrow$

$\exists$ Noether current $J_\mu(x)$ $\quad (\partial_\mu J_\mu = 0$ w/ EoM.)

$\Downarrow$

Ward-Takahashi identity

$$\partial_\mu \langle J_\mu(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \sum_{i=1}^{n} \delta(x-x_i) \langle \mathcal{O}_1(x_1) \cdots \delta\mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle .$$

WT identity is the most essential ingredient for symmetries in QFT.
Important properties (Selection rule, Nambu-Goldstone thm, Anomaly matching ...) come out solely from WT identity,
so we do not need Lagrangian's invariance once we get WT id.

$\Big\{$ Extract its essence

Symmetry in QFT = Topological operators

(Gaiotto, Kapustin, Seiberg, Willet '14, ----)

This leads to the vast generalization for the notion of global symmetry.

Conventional symmetry (= 0-form invertible symmetry)
codim-1 top. op. w/ group-type fusion rule

$\Rightarrow$ Higher-form / higher-group symmetry
- Higher-codim. top. operator w/ group-like fusion rule

[Gaiotto, et.al. '14; Sharpe '15, Cordova, Dumitrescu, Intriligator, YT, Ünsal '19]

$\Downarrow$

Non-invertible sym. in 2-d.
- codim-1 top. op. w/ fusion category

[Verlinde '88; Bhardwaj, Tachikawa '17; Thorngren, Wang '19] ---

$\Rightarrow$ Non-inv. sym. in general dim.

[Nguyen, YT, Ünsal '21; Koide, Nagoya, Yamaguchi '21; Choi, Cordova, Hsin, Lam, Shao '21; Kaidi, Ohmori, Zheng '21, ---]

* In this lecture, we will only use higher-form symmetry, especially for YM.

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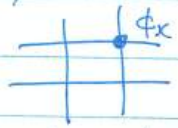
* Explicit construction of topological operator for classical statistical lattice model.

Consider real ϕ^4-theory $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!}\lambda \phi^4$.
$\mathbb{Z}_2$ symmetry : $\phi \mapsto -\phi$.

If [Top op. = Sym.], we should be able to construct the corresponding topological operator. ⇝ How?

Convenient trick (for electric symmetry) is to consider the lattice model & to introduce the background gauge field.

[* One can construct it in the same way also in the continuum at least formally.
~~Here~~, As long as the symmetry has the on-site lattice realization, this always works.]



- lattice model

$$S_{lat}[\phi] = K \sum_x \sum_\mu \phi_{x+\hat{\mu}} \phi_x + \sum_x V(\phi_x)$$

↑ lattice hopping term. ↙ local potential at site x.

- Background gauge field.

$\mathbb{Z}_2$ sym : $\phi_x \mapsto -\phi_x$ is an on-site sym.
We can introduce the $\mathbb{Z}_2$ lattice gauge field (= $\mathbb{Z}_2$ link variables) to promote the global invariance to the local invariance :

$$S_{lat}[\phi, A] = K \sum_{x,\mu} \phi_{x+\hat{\mu}} e^{iA_{x,\mu}} \phi_x + \sum_x V(\phi_x).$$

$$A_{x,\mu} \in \{0, \pi\} \simeq \mathbb{Z}_2.$$

Background $\mathbb{Z}_2$ gauge inv.

$$\begin{cases} A_{x,\mu} \mapsto A_{x,\mu} + \lambda_{x+\hat{\mu}} - \lambda_x \\ \phi_x \mapsto e^{i\lambda_x} \phi_x \end{cases}$$

$(\lambda_x \in \{0, \pi\})$.

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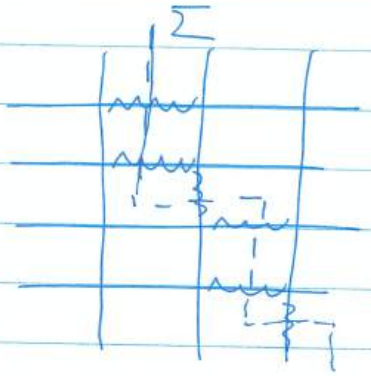
We define the $\mathbb{Z}_2$ symmetry operator $U_\pi(\Sigma)$ defined on codim-1 surface Σ as followes:



- Regard Σ as a codim-1 surface of "dual lattice"
- We set

$$A_{x,\mu} = \begin{cases} \pi & \text{(if the link } (x,\mu) \text{ goes across } \Sigma) \\ 0 & \text{(otherwise)} \end{cases}$$

$$= \pi \times (\text{Poincaré Dual of } \Sigma) \quad (\pi\, PD(\Sigma))$$

- $\langle \mathcal{O}\, U(\Sigma) \rangle$ ($\mathcal{O}$: some local operators written in term of ϕ_x (such as ϕ_x itself, ϕ_x^2, $\phi_x\phi_y$, etc.))

$$:= \frac{1}{Z[0]} \int \mathcal{D}\phi\, e^{-S_{lat}[\phi,\, A=\pi PD[\Sigma]]}\, \mathcal{O}.$$

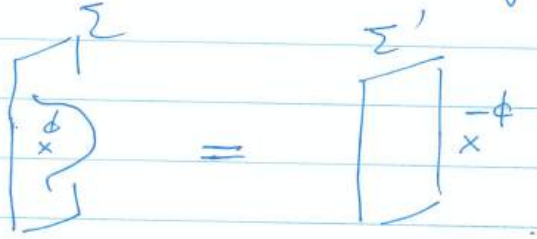
i.e. $U(\Sigma)$ is a defect operator.

Topologicalness:

Background $\mathbb{Z}_2$ gauge transformation changes the shape of Σ.
Due to its gauge invariance,

$$\langle U(\Sigma) \cdots \rangle = \langle U(\Sigma') \cdots \rangle$$

unless the deformation goes across other operator.



When the deformation crosses other operators (say ϕ_x), then $\phi_x \mapsto -\phi_x$.

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① $\mathbb{Z}_N$ 1-form symmetry of $SU(N)$ Yang-Mills theory.

Let's consider $SU(N)$ YM:

$$S = \frac{1}{g^2}\int \mathrm{tr}(F\wedge *F) \qquad \left(+ i\frac{\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F)\right)$$

(if 4d, we can add θ-term.)

$a = \sum_{\mu,i} a_\mu^i T^i dx^\mu$ is the $SU(N)$ gauge field

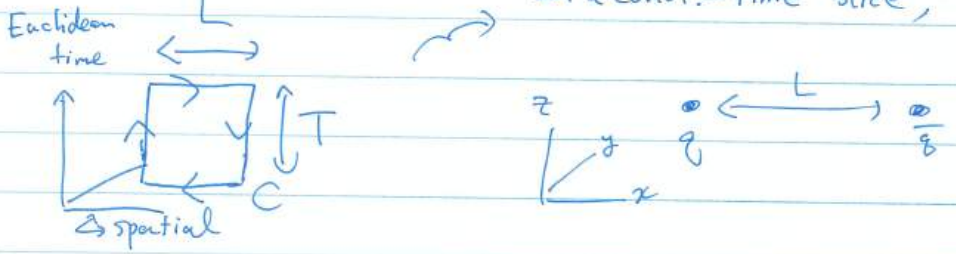
$F = da + ia\wedge a$: $SU(N)$ field strength.

$SU(N)$ gauge trans. $a \mapsto a^g = g a g^{-1} - i g dg^{-1}$.

The basic gauge-inv. observables are constructed from Wilson loops

$$W(C) = \frac{1}{N}\mathrm{tr}\left[\mathcal{P} e^{i\int_C a}\right],$$

which describes the worldline of the test quark:



$$\langle W(C)\rangle = \exp\left[-T\times \underbrace{V(L)}\right]$$

inter-quark potential.

$$V(L) \overset{L\to\infty}{\sim} \begin{cases} \sigma L & \text{(confined : } q\,\bar q\text{)} \\ \text{const.} & \text{(Higgs : } q\ \ \bar q\text{)} \\ \frac{1}{L} & \text{(Coulomb) gapless.} \end{cases} \Big\} \text{gapped}$$

YM theory has the $\mathbb{Z}_N$ 1-form symmetry;

(confined phase $\longleftrightarrow$ gapped & unbroken $\mathbb{Z}_N^{(1)}$ sym.
Higgs phase $\longleftrightarrow$ gapped & spontaneously broken $\mathbb{Z}_N^{(1)}$ sym.

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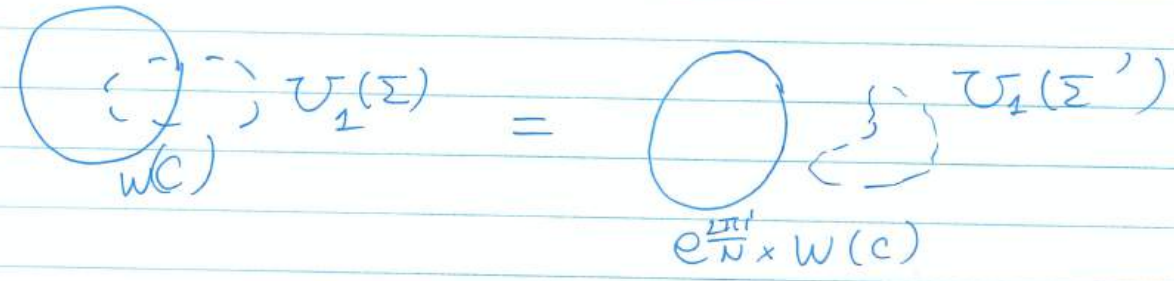
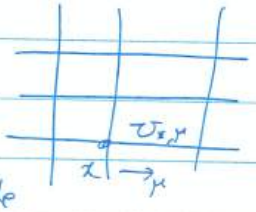
$\mathbb{Z}_N$ 1-form sym: (line in 3d, surface in 4d, etc.)

- codim-2 topological operator $U_k(\Sigma)$ w/ $k \sim k+N$
- $U_k(\Sigma)\, U_{k'}(\Sigma) = U_{k+k'}(\Sigma)$
- 

Lattice

◎ ∧ Construction.

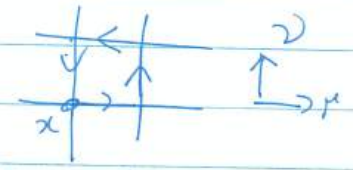
Wilson's lattice gauge theory:



- $U_{x,\mu}\ (= \mathcal{P} e^{i \int_{\mathrm{link}(x,\mu)} a})\ \in SU(N)$: fund. variable

  $SU(N)$ gauge trans $U_{x,\mu} \mapsto g_x^{-1} U_{x,\mu}\, g_{x+\hat{\mu}}$

- $U_{x,\mu\nu} := U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U_{x+\hat{\nu},\mu}^{-1}\, U_{x,\nu}^{-1}$



- $S_{\mathrm{lat}}[U_\ell] = -\beta \sum_{x,\mu\nu} \left(\mathrm{tr}(U_{x,\mu\nu}) + \mathrm{tr}(U_{x,\mu\nu}^\dagger) \right)$.

This lattice action satisfies

- × local $SU(N)$ gauge invariance
- × (hyper-)cubic rotational invariance.

Since $\mathrm{tr}(F_{\mu\nu}^2)$ is the lowest-dimensional operator satisfying these properties, this gives ∧ (one of) the lattice regularizati of YM theory.

To construct the $\mathbb{Z}_N$ 1-form sym. operator, we should couple the $\mathbb{Z}_N$ 2-form background gauge field (= $\mathbb{Z}_N$ plaquette variable)

$$S_{\mathrm{lat}}[U, B_P] = -\beta \sum_P \left(e^{-iB_P}\, \mathrm{tr}(U_{\partial P}) + e^{iB_P}\, \mathrm{tr}(U_{\partial P}^\dagger) \right)$$

$(B_P \in \frac{2\pi}{N}\{0, 1, \cdots, N-1\} \simeq \mathbb{Z}_N)$.

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e^{-iB_p} $tr(U_{\partial p})$:

$\mathbb{Z}_N$ 1-form gauge invariance

$$\begin{cases} B_p \mapsto B_p + \frac{2\pi}{N}(d\lambda)_p \\ U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell \end{cases}$$

$U_k(\Sigma)$ is equiv. to set

$$B = \frac{2\pi}{N} k \, PD(\Sigma).$$

The above 1-form gauge invariance ensures the topologicalness of $U(\Sigma)$.

't Hooft Anomaly.

As we've seen, when the QFT has a global symmetry, we can couple it with the background gauge field A:

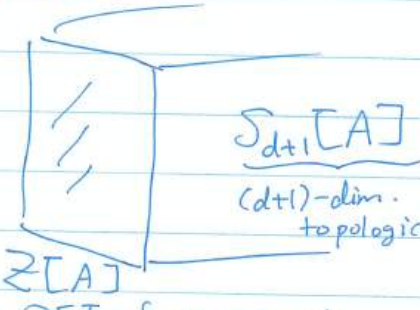
$$Z[A] = \int \mathcal{D}\Phi \, \exp\left(-S[\Phi, A]\right).$$

The background gauge invariance ensures the topologicalness of the symmetry operator, but it may have a mild violation :

$$Z[A^\lambda] = \underbrace{e^{i \mathcal{A}_d(\lambda, A)}}_{\text{'t Hooft anomaly.}} Z[A].$$

(gauge trans.)

Anomaly-inflow



The combined system achieves the gauge invariance

$$Z[A]\, e^{i S^{top}_{d+1}[A]}.$$

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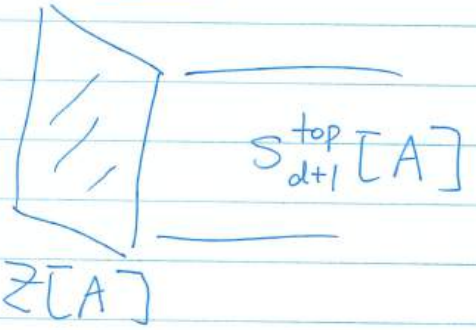
② Anomaly matching condition.



The top. action (such as Chern-Simons $\frac{k}{4\pi}\int A dA$) only has the discrete label.

~~Anomaly~~

⇒ So does the anomaly, since A is just the bdry gge violation.

The gauge-inv. of the combined system then shows the anomaly of d+1 QFT does not change under Local & Symmetric deformations.

In particular:

't Hooft anomaly is inv. under the RG flow.

Thus, the 't Hooft anomaly should be reproduced by the ground states (& associated gapless excitations).

③ Applications to the θ-vacua of 4d YM theory
(Gaiotto, Kapustin, Komargodski, Seiberg '17)

4d SU(N) gge field has the topological sectors, characterized by

$$Q_{top} = \frac{1}{8\pi^2}\int \mathrm{tr}(F\wedge F) \in \mathbb{Z}.$$

Thus

$$Z_\theta = \int \mathcal{D}a \exp(-S_{YM} + i\theta Q_{top})$$

has the 2π periodicity: $Z_{\theta+2\pi} = Z_\theta$.

⇝ As the quantum mechanical system, YM at θ & YM at $\theta+2\pi$ are unitary equivalent.

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If we introduce the background $\mathbb{Z}_N$ 2-form gauge field B, the instanton number is shifted by $\frac{1}{N}$: (van Baal '82)

$$Q_{\text{top}}[B] = \underbrace{-\frac{N}{8\pi^2}\int B\wedge B}_{\frac{1}{N}\text{-quantized}} + \mathbb{Z}$$

* Its convenient derivation in continuum can be found in Kapustin, Seiberg '14.
* This is also rigorously proven on the lattice w/ admissibility condition by Abe, Morikawa, Onoda, Suzuki, YT '23, which extends Lüscher '82.

* Rough idea (or good analogy)

Consider particle on $S^1 \ni e^{ix}$

which has the winding number $\frac{1}{2\pi}\int dx \in \mathbb{Z}$.

If we ~~introduce~~ the $\mathbb{Z}_N$ background gauge field A for $x \mapsto x + \frac{2\pi}{N}$, the b.c. becomes the twisted one: $x(\tau+L) \sim x(\tau) + \underbrace{\frac{2\pi}{N}}_{\int A} \times k$

Then

$$\frac{1}{2\pi}\int dx = \frac{1}{2\pi}(x(L)-x(0)) = \underbrace{\frac{k}{N}}_{\frac{1}{2\pi}\int A} + \mathbb{Z}$$

This proves the (generalized) 't Hooft anomaly

$$Z_{\theta+2\pi}[B] = e^{-i\frac{N}{4\pi}\int B\wedge B} Z_\theta[B].$$

[More precisely, this is the one called {global inconsistency, or coupling-space anomaly.
For its matching condition, see e.g., Cordova, Freed, Lam, Seiberg '19; Kikuchi, YT '17, YT, Sulejmanpasic '18,]

$\Rightarrow$ If YM shows confinement for any θ (i.e. $\mathbb{Z}_N^{(1)}$ is always unbroken), there must be a quantum phase transition as $\theta \to \theta+2\pi$.

(cf. 't Hooft '79)

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§2. Semiclassics for QM of a particle on S^1.

We consider a particle on $x \sim x + 2\pi \in S^1$, with the Lagrangian

$$S = \int d\tau \left\{ \frac{1}{2}\dot{x}^2 - g\cos(Nx) \right\} \pm i\frac{\theta}{2\pi}\int dx .$$

(↑ This turns out to be a good toy model for YM on $\mathbb{R}^2 \times T^2_{\text{twisted}}$)

$\mathbb{Z}_N$ Symmetry : $x \mapsto x + \frac{2\pi}{N}$.
(This is $\mathbb{Z}_N$ instead of $\mathbb{Z}$ since $x \overset{N\text{-times operation}}{\mapsto} x + 2\pi \sim x$ gauge equiv.)

When $\theta \in \pi\mathbb{Z}$, there also exists $\mathbb{Z}_2$ charge conjugation (or parity):
$x \mapsto -x$.
(Under this op. $\theta \mapsto -\theta$. Since $\theta \sim \theta + 2\pi$, the inv. points are $\theta = 0$ or π)

⑩ (Generalized) anomaly

$$Z_\theta[A] = \int \mathcal{D}x \exp\left[-\int d\tau \left\{ \frac{1}{2}(\dot{x} + A_\tau)^2 - g\cos(Nx) \right\} + i\frac{\theta}{2\pi}\int (dx + A) \right]$$

As we change $\theta \mapsto \theta + 2\pi$, we find

$$Z_{\theta+2\pi}[A] = \int \mathcal{D}x \exp\left[-\int d\tau \left(\frac{1}{2}|\dot{x} + A_\tau|^2 - g\cos(Nx) \right) + i\left(\frac{\theta}{2\pi} + 1\right)\int (dx + A) \right]$$

($\int (dx + A)$: 2π-quantized)

$$= e^{i\int A}\, Z_\theta[A] .$$

⇝ There must be a phase transition (= level crossing of the ground state) as we change $\theta \mapsto \theta + 2\pi$. ("i" disappears.)

Hamiltonian viewpoint: In real-time Lag. $S_{\text{real-time}} = \int \left\{ \frac{1}{2}\dot{x}^2 + g\cos(Nx) + \frac{\theta}{2\pi}\dot{x} \right\} dt$.
Canonical momentum $p = \frac{\partial L_{\text{real time}}}{\partial \dot{x}} = \dot{x} + \frac{\theta}{2\pi}$. ∴ $H_\theta = p\dot{x} - L = \frac{1}{2}\left(p - \frac{\theta}{2\pi}\right)^2 - g\cos(Nx)$.
This QM system is defined by H_θ & $\mathcal{H} = L^2(S^1)$.

$Z_{\theta+2\pi} = Z_\theta$ can be understood as $H_{\theta+2\pi} = e^{+ix} H_\theta e^{-ix}$ (unitary operator).

$\Rightarrow$ $|GS \text{ at } \theta + 2\pi\rangle = e^{+ix}\,|GS \text{ at } \theta\rangle$.

$|GS \text{ at } \theta + 2\pi\rangle$: $\mathbb{Z}_N$ charge $\ell + 1$.
e^{+ix} : $\mathbb{Z}_N$ charge 1.
$|GS \text{ at } \theta\rangle$: $\mathbb{Z}_N$ charge ℓ.

∴ Level crossing is necessary.
(If the GS wave func. has the specific $\mathbb{Z}_N$ charge) ⑩

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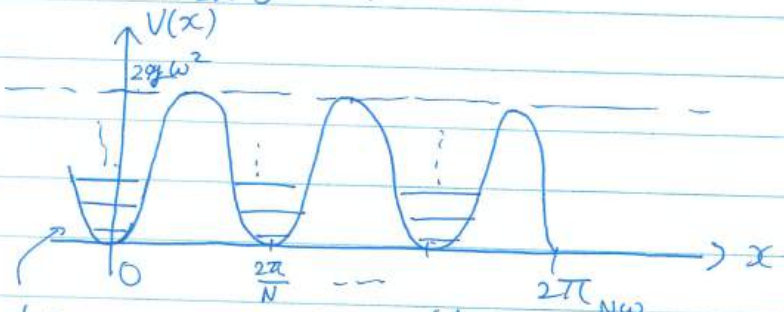
Let's confirm this symmetry prediction by the explicit computation of the ground-state energies.

⇒ semiclassical calculus (WKB approximation)

$$S = \int d\tau \left\{ \frac{1}{2}\dot{x} + \underbrace{\frac{g}{\omega^2}(1-\cos(Nx))}_{V(x)\ :\ \text{here we set an offset so that } V(x)\geq 0} \right\} - i\frac{\theta}{2\pi}\int dx .$$



$$V(x\simeq 0) \approx \frac{N^2 g^2}{2} x^2$$

perturbative spectrum $E_\ell = \left(\frac{1}{2}+\ell\right)\sqrt{N^2 g^2}\,\omega^2 \quad (\ell \geq 0)$,

with the harmonic-oscillator wave func. localized around $x = \frac{2\pi}{N}k$.

At this stage, $\mathbb{Z}_N$ is broken.

We denote this $\mathbb{Z}_N$-broken classical vacua as $|k\rangle$. $(k=0,1,\cdots,N-1)$
(i.e. $\ell = 0$)

However, these classical vacua $|k\rangle$ are not eigenstates of the Hamiltonian quantum mechanically.

To see this, let's consider

$$\langle k'|\, e^{-TH}\, |k\rangle = \int_{\substack{x(T)=\frac{2\pi}{N}k' \\ x(0)=\frac{2\pi}{N}k}} \mathcal{D}x \; \exp\left[-\int_0^T d\tau \left\{ \frac{1}{2}\dot{x}^2 + \frac{g}{\omega^2}(1-\cos(Nx)) - i\frac{\theta}{2\pi}\dot{x} \right\} \right].$$

In particular, we'd like to evaluate the case $k \neq k'$ mod N.

⇒ Bogomolnyi trick

We put an off-set $V(x) \geq 0$ & $\min_x V(x) = 0$.

To find the solution from one minimum x_1 to the other x_2, we rewrite

$$S = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V(x)\right) = \int d\tau\, \frac{1}{2}\left(\dot{x} \mp \sqrt{2V(x)}\right)^2 \pm \underbrace{\int d\tau\, \dot{x}\sqrt{2V}}_{\int_{x_1}^{x_2} dx \sqrt{2V(x)}\ \text{determined by B.C.}}$$

∴ Minimization of the Euclidean action is achieved by

$$\dot{x} = \pm\sqrt{2V(x)}.$$

↑ The sign should be chosen so that $S \geq 0$.

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The tunneling solution $|k=0\rangle \Rightarrow |k=1\rangle$

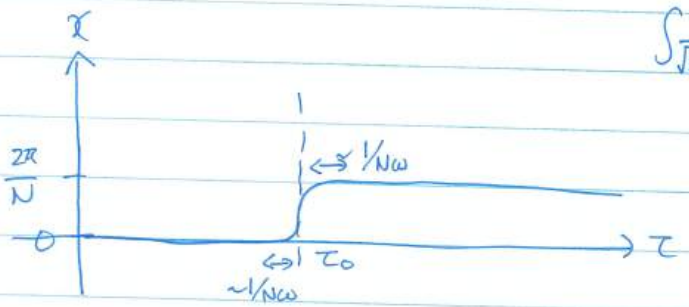
$$\int_0^T d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}(1-\cos(Nx)) \right)$$

$$= \int_0^T d\tau \underbrace{\frac{1}{2}\left(\dot{x} - \sqrt{\frac{2}{\omega^2}(1-\cos(Nx))}\right)^2}_{\geq 0} + \underbrace{\int_0^{\frac{2\pi}{N}} dx \sqrt{\frac{2}{\omega^2}(1-\cos(Nx))}}_{=: \omega \cdot S_I}$$

$$\geq \omega S_I .$$

The inequality is saturated iff $\frac{dx}{d\tau} = \omega\sqrt{2(1-\cos(Nx))}$.

$$\int \frac{dx}{\sqrt{2(1-\cos(Nx))}} = \omega(\tau - \tau_0).$$



Instantaneously jumps from $\frac{2\pi}{N}\cdot 0 \Rightarrow \frac{2\pi}{N}\cdot 1$.
"Instanton".

When we formally set $T \to \infty$, the system is translationally inv.
$\Rightarrow$ the location of τ_0 is arbitrary for $0 \leq \tau_0 \leq T$.
(= classical moduli).

Within the one-instanton approximation

$$\langle k=1| e^{-TH} |k=0\rangle = \int_{\substack{x(0)=0 \\ x(T)=\frac{2\pi}{N}}} \mathcal{D}x \; e^{-\int_0^T d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}(1-\cos(Nx))\right) + i\frac{\theta}{2\pi}\int dx}$$

$$= \underbrace{\omega T}_{\substack{\text{(approximate) moduli} \\ \text{integration}}} \times \underbrace{A}_{O(1)\text{ factor}} \times e^{-\omega S_I + i\frac{\theta}{N}}$$

For the correct evaluation of the prefactor,
we need a careful estimation for the functional determinant with the instanton background.
For QM case, Gelfand-Yaglom formula is useful for this purpose.

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Within the one-instanton approximation, the $N \times N$ matrix $\langle k' | e^{-TH} | k \rangle$ looks like

$$\left(\langle k' | e^{-HT} | k \rangle\right)_{k,k'=0,1,\cdots,N-1} = \begin{pmatrix} 1 & A\omega T e^{-\omega S_I + i\frac{\theta}{N}} & & & A\omega T e^{-\omega S_I - i\frac{\theta}{N}} \\ A\omega T e^{-\omega S_I - i\frac{\theta}{N}} & 1 & A\omega T e^{-\omega S_I + i\frac{\theta}{N}} & & \\ & A\omega T e^{-\omega S_I - i\frac{\theta}{N}} & 1 & \ddots & \\ & & \ddots & \ddots & A\omega T e^{-\omega S_I + i\frac{\theta}{N}} \\ A\omega T e^{-\omega S_I + i\frac{\theta}{N}} & & & A\omega T e^{-\omega S_I - i\frac{\theta}{N}} & 1 \end{pmatrix}$$

We can diagonalize this matrix by

$$|\tilde{l}\rangle = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \omega^l \\ \vdots \\ \omega^{(N-1)l} \end{pmatrix} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{kl} |k\rangle \qquad (\omega = e^{\frac{2\pi i}{N}})$$

$\rightsquigarrow$ eigenvalue $1 + A\omega T e^{-\omega S_I} \left(e^{i\frac{\theta + 2\pi l}{N}} + e^{-i\frac{\theta + 2\pi l}{N}} \right)$

$$= 1 + 2A\omega T e^{-\omega S_I} \cos\left(\frac{\theta + 2\pi l}{N}\right)$$

$$\cong \exp\left[-T\left(-2A\omega e^{-\omega S_I} \cos\left(\frac{\theta + 2\pi l}{N}\right)\right)\right]$$

Interpretation

Since we diagonalize e^{-TH}, the eigenvalues should take the form of e^{-TE_l}.

In the one-instanton approximation, we should assume

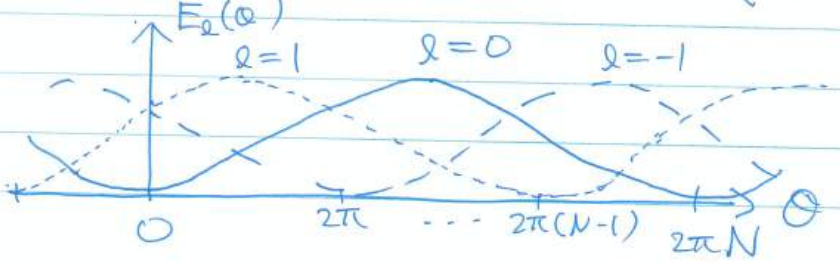
$$2A\omega T e^{-\omega S_I} \ll 1.$$

(while $\omega T \gg 1$)

and thus $e^{-TE_l} \approx 1 - TE_l$. (higher-order terms in TE_l are too small)

The energy eigenvalue for $|\tilde{l}\rangle$ is thus

$$E_l(\theta) = -2A\omega e^{-\omega S_I} \cos\left(\frac{\theta + 2\pi l}{N}\right).$$



Level crossing !!

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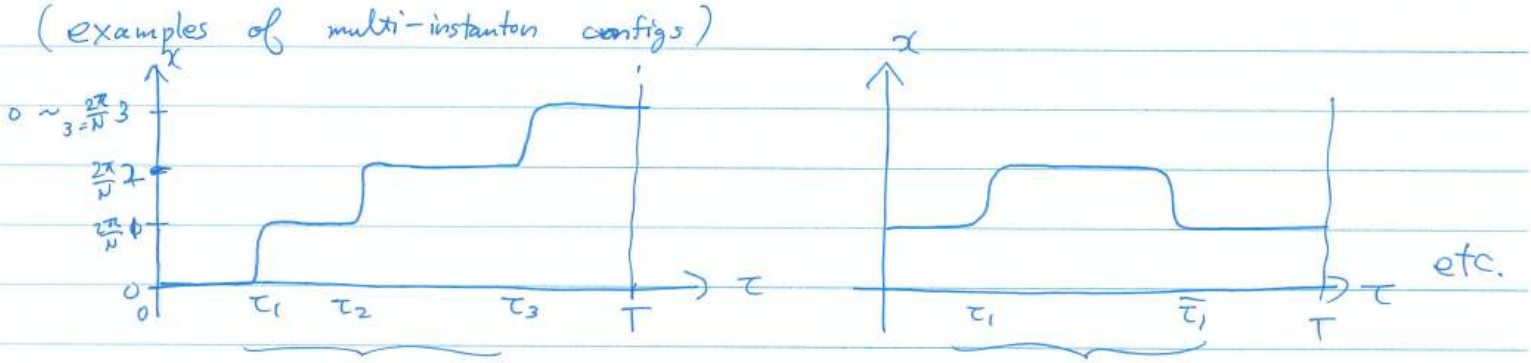
⑪ Dilute-instanton-gas approximation for the partition function.

Let us evaluate

$$Z_\theta = \mathrm{tr}\left[e^{-TH_\theta}\right] = \int_{x(0)\sim x(T)} \mathcal{D}x \; e^{-\int d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{g^2}(1-\cos(Nx))\right) + i\frac{\theta}{2\pi}\int dx}$$

We are particularly interested in the properties of the ground states, so we consider $\omega T \gg 1$.
Here, we do "not" necessarily assume $\omega T e^{-\omega S_I} \ll 1$, so we have to take into account the multi-instanton effect.

(examples of multi-instanton configs)



N instantons

$$S = N\omega S_I + i\theta + \mathcal{O}\left(e^{-\kappa\omega|\tau_i-\tau_j|}\right)$$

instanton – anti-instanton

$$S = 2\omega S_I + \mathcal{O}\left(e^{-\kappa\omega|\tau-\bar{\tau}|}\right)$$

⇝ We can regard the configurations as the ensembles of instantons & anti-instantons.
Since each instanton goes to classical vacua w/ exponentially fast convergence, we may neglect the "interaction" between multi-instantons.
(Dilute gas approximation).

$$Z_\theta = \sum_{n,\bar{n}\geq 0} \frac{1}{n!\,\bar{n}!}\left(A\omega T e^{-\omega S_I + i\frac{\theta}{N}}\right)^n \left(A\omega T e^{-\omega S_I - i\frac{\theta}{N}}\right)^{\bar{n}} \times \underbrace{\delta_{n-\bar{n}\in N\mathbb{Z}}}_{\text{periodic B.C. } x(T)\sim x(0) \text{ requires } \#(\text{instanton}) - \#(\text{anti-instanton}) \text{ quantize in } N.}$$

$$= \sum_{\ell=0}^{N-1} \sum_{n,\bar{n}\geq 0} \frac{e^{\frac{2\pi i}{N}\ell(n-\bar{n})}}{n!\,\bar{n}!}\left(A\omega T e^{-\omega S_I + i\frac{\theta}{N}}\right)^n \left(A\omega T e^{-\omega S_I - i\frac{\theta}{N}}\right)^{\bar{n}}$$

$$= \sum_{\ell=0}^{N-1} \exp\left[-T\underbrace{\left(-2A\omega e^{-\omega S_I}\cos\left(\frac{\theta+2\pi\ell}{N}\right)\right)}_{=E_\ell(\theta)}\right].$$

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§3. Semiclassics for 4d SU(N) YM on $\mathbb{R}^2 \times T^2_{\text{twisted}}$.

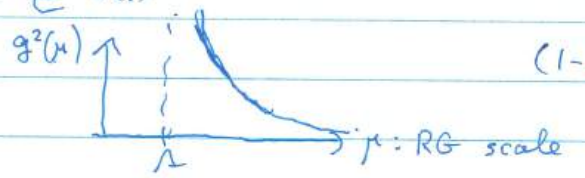
4d YM theory

$$S = \frac{1}{g^2}\int \mathrm{tr}(F\wedge *F) + i\frac{\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F)$$

is asymptotically free : It is classically scale invariant, and $g(\mu)$ runs (quantum mechanically)

$$\left(\frac{\Lambda}{\mu}\right)^{\frac{11}{3}N} = e^{-\frac{8\pi^2}{g^2(\mu)}}$$

Λ : strong scale



(1-loop RG).

⇝ Theory becomes strongly coupled at $E \sim \Lambda$,
and it is difficult to predict its low-energy properties.

Belief : ① Non-zero mass gap ② Confinement. $\Rightarrow \langle W(C)\rangle = e^{-\sigma \times \mathrm{Area}(C)}$.

Why do we believe so? Many evidences!

① Experimental facts : No one doubts QCD in the Standard Model describes the fundamental theory for nuclear/hadron physics.
We have never observed colored particles as asymptotic states ⇐ Confinement explains this phenomenological fact.

② Lattice ~~QCD~~ Gauge Theory : Lattice provides the manifestly gauge-inv. regularizati for YM/QCD.
Lattice strong coupling expansion "proves" confinement. (Wilson)
Numerically, no phase transition is observed between lattice weak couplings and lattice strong couplings. (Creutz)

③ Supersymmetric version : $\mathcal{N}=2$ super YM has the monopole/dyon points in its moduli, and the soft SUSY breaking $\mathcal{N}=2 \Rightarrow \mathcal{N}=1$ prefers their condensation.
⇐ Explicit achievement of the "dual superconductivity" picture! (Seiberg, Witten)

④ Holography : In type-IIA/IIB superstrings, maximally supersymmetric U(N) YM lives on ND branes. (When $N \gg 1$ & $g^2N \gg 1$).
4d Non-SUSY YM can be realized with N D4 branes on $\mathbb{R}^4 \times S^1 \times \mathbb{R}^5$ (D4 extend) w/ anti-periodic b.c. for S^1.
Analysis of the gravity side shows confinement. (Witten).

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We here try to give another evidence by studying YM on $\mathbb{R}^2 \times T^2$ with the 't Hooft twisted B.C. (YT, Ünsal '22)

[cf. van Baal '88 ; -, Yamazaki, Yonekura '17, Cox, Poppitz, Wandler '21, ...] (for $\mathbb{R} \times T^3$)

① YM theory on a compactified T^2.

Before considering T^2, let's briefly review the case of YM on $\mathbb{R}^3 \times S^1$.

Why do we want to compactify?

L: circumferance of S^1 ⇝ Typical energy scale $E = \frac{2\pi}{L}$.

If $E = \frac{2\pi}{L} \gg \Lambda$, ~~the typical energy~~ $g^2(E)$ is small, and we may be able to perform the weak-coupling analysis for asymptotically free theories.

On $\mathbb{R}^3_{\vec{x}} \times S^1_{\tau = x_4}$, a_4 can be used to give a local operator in 3d effective theory by considering the Wilson loop wrapping S^1 (= Polyakov loop)

$$P(\vec{x}) = \mathcal{P} \exp\left(i \int_0^L d\tau \, a_4(\vec{x}) \right).$$

$P(\vec{x}) \in SU(N)$, and, under the 3d gauge transformation $g(\vec{x})$, $P(\vec{x}) \mapsto g^\dagger(\vec{x}) P(\vec{x}) g(\vec{x})$.

Using this, it is convenient to diagonalize $P(\vec{x})$ (Polyakov gauge)

$$P(\vec{x}) = \begin{pmatrix} e^{i\phi_1(\vec{x})} & & 0 \\ & e^{i\phi_2(\vec{x})} & \\ 0 & & \ddots \\ & & & e^{i\phi_N(\vec{x})} \end{pmatrix}.$$

Here, we require $\phi_1 + \cdots + \phi_N = 0$ ($\because P \in SU(N)$ requires this is true in mod 2π. Here, we further require this is true literally.)

so $a_4 = \frac{1}{L} \begin{pmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_N \end{pmatrix}$.

Classical action $\frac{1}{g^2} \mathrm{tr}(F_{\mu\nu}^2)$ does not give the potential for $\vec{\phi}$, i.e. $\vec{\phi}$ is a classical moduli.

At the 1-loop level, its potential is generated as

$$V_{1\text{-loop}}(P) \sim -\frac{1}{L^4} \sum_{n=1}^{\infty} \frac{|\mathrm{tr}(P^n)|^2}{n^4}.$$

(Gross, Pisarski, Yaffe).

- This is inv. under $P \mapsto e^{\frac{2\pi i}{N}} P$. ($\mathbb{Z}_N$ 0-form) center symmetry).
- $\langle P \rangle = e^{\frac{2\pi i}{N} k} 1_N$ (center-broken vacua) are preferred.

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Intuitive explanation

Here, I do not give the derivati of the GPY potential, but instead, let me give an intuitive explanation why center-broken points are preferred.

Let's consider the Kaluza-Klein (or Matsubara) expansion

$$a_{\substack{I \\ 1\sim 3}}^{\mu}(x,\tau) = \sum_n e^{\frac{2\pi i}{N L} n\tau}\, \tilde{a}_I^{(n)\mu}(x).$$

In the classical Lagrangian $\mathrm{Tr}((F_{\mu\nu})^2) \supset \mathrm{Tr}((F_{I4})^2)$, which gives

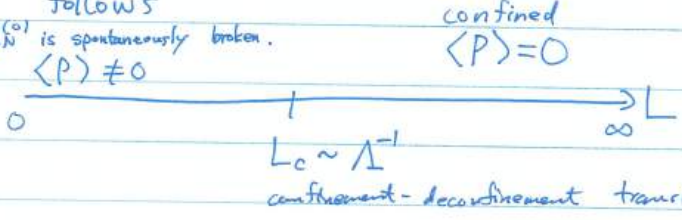
$$\begin{cases} Q_4\, a_I \Rightarrow \left\{\frac{2\pi n}{L} i\right\} (\tilde{a}_I^{(n)}) \\ [a_I, a_4] \Rightarrow \sum_{ij=1}^{N} \frac{\phi_i - \phi_j}{L}\, a_I^{ij} \end{cases} \qquad \left(a_4 = \frac{1}{L}\begin{pmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_N \end{pmatrix}\right)$$

and thus produces the 3d mass term

$$\left(\frac{2\pi}{L} n + \frac{\phi_i - \phi_j}{L}\right)^2 \left|\tilde{a}_I^{(n)ij}\right|^2.$$

When $\vec{\phi} = \vec{0}$, the number of massless modes is $N^2 - 1$, which is maximal. $\Rightarrow$ Largest entropy & preferred!

As a result, the phase diagram for YM on $\mathbb{R}^3 \times S^1$ should look as follows



Since we want to study the properties of the confinement states, the naive compactification does not fit our purpose.

(* YM on $\mathbb{R}^3 \times S^1_{\text{small}}$ = YM at high temperatures.
Thus, ~~How its study~~ studying its property is quite valuable in the context of the heavy-ion collision experiments.)

(* If there exists n_f adjoint fermions with the periodic b.c. (non-thermal one), the GPY potential is flipped:
$V^{(GPY)}(P) \sim \frac{n_f - 1}{L^4} \sum_{n\geq 1} \frac{\mathrm{tr}(P^n)}{n^4}$. $\rightsquigarrow P \propto (1, \omega, \cdots, \omega^{N-1})$ (center-symmetric point) is preferred.
This actually gives the nice calculable setup for the confinement physics (Unsal, Yaffe, Shifman, Poppitz, ... '07
cf. Davies, Hollowood, Khoze, Mattis '00)

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- Classical vacuum for $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux & Perturbative spectrum.

What we want to achieve is

confined weakly-coupled ⟵ smoothly connected ⟶ confined strongly-coupled

0 ⟶ ∞ : L: compactification scale

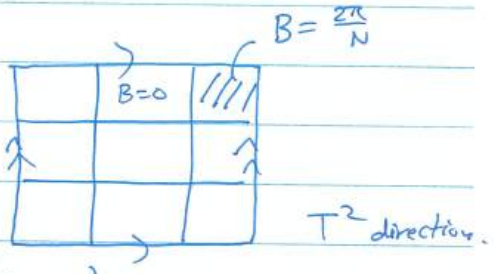
Claim: This is achieved for YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux $\int_{T^2} B = \frac{2\pi}{N}$.

Let's first find the classical vacuum for YM w/ this twisted B.C., and it is convenient to take the Wilson (lattice regularization

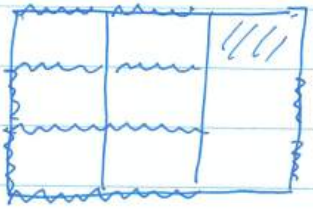
$$S_W[U_\ell, B_p] = -\beta \sum_p \left(e^{-iB_p} \mathrm{tr}(U_{\partial p}) + \text{c.c.} \right)$$

$\int_{T^2} B = \frac{2\pi}{N}$ can be realized by setting

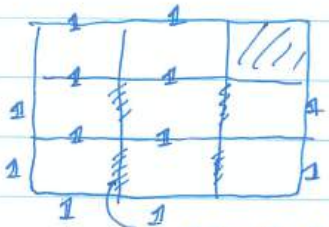
$$B_p = \begin{cases} 0 & \text{for unshaded plaquettes} \\ \frac{2\pi}{N} & \text{for the shaded one (i.e. right-top one)} \end{cases}$$

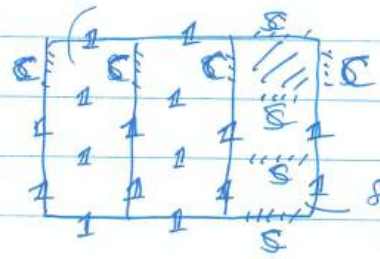


If $e^{-iB_p} U_{\partial p} = \mathbb{1}_N$ for all the plaquettes, S_W is minimized.
(This is a sufficient condition, but not necessary. We shall see this is achieved below.)

① 

Take the complete axial gauge so that $U_\ell = \mathbb{1}$ if $\ell = nn$.

② 

③ 

④ ← For $e^{-iB_p} U_p = \mathbb{1}$, $SC = e^{\frac{2\pi i}{N}} CS$.

This is unique up to gauge trans.: $C = \omega^{\frac{N-1}{2}} \begin{pmatrix} 1 & & \\ & \omega & \\ & & \ddots \\ & & & \omega^{N-1} \end{pmatrix}$, $S = \omega^{\frac{N-1}{2}} \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$. SU(N) clock & shift matrix.

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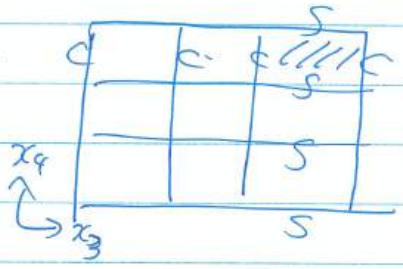
Then, the Polyakov loops along the 3rd & 4th directions are given by

$$\begin{cases} P_3 = \mathcal{P}\, e^{i\int a_3 dx_3} = S \\ P_4 = \mathcal{P}\, e^{i\int a_4 dx_4} = C \end{cases}$$



$\Rightarrow$ $\mathrm{tr}\left(P_3^{n_3} P_4^{n_4}\right) = 0$ unless $n_3 = 0$ & $n_4 = 0$ mod N.

$\mathbb{Z}_N \times \mathbb{Z}_N$ center symmetry is unbroken for this classical vacuum.

This is quite suggestive & this setup is promissing for the adiabatic continuity !!

Gapped

• Perturbative spectrum

Moreover this setup gives the perturbative mass gap for 2d gluons, $a_{I(=1,2)}$.

As the classical vacuum is given by $P_3 = S$ & $P_4 = C$, the 2d adjoint field (after suitable redef. of fields so that $a_{3,4} = 0$) obeys

$$\begin{cases} a_I(x_3 + L_3, x_4) = S^{-1}\, a_I(x_3, x_4)\, S \\ a_I(x, x_3, x_4 + L_4) = C^{-1}\, a_I(x_3, x_4)\, C. \end{cases}$$

To perform the Kaluza-Klein reduction with this B.C., it is convenient to represent the generators of the $SU(N)$ Lie algebra as follows

$$J_{(p_3,p_4)} = \omega^{-\frac{p_3 p_4}{2}} C^{-p_3} S^{p_4},$$

so that

$$a_I = \sum_{(p_3,p_4)\neq 0} a_I^{(p_3,p_4)}\, J_{(p_3,p_4)}.$$

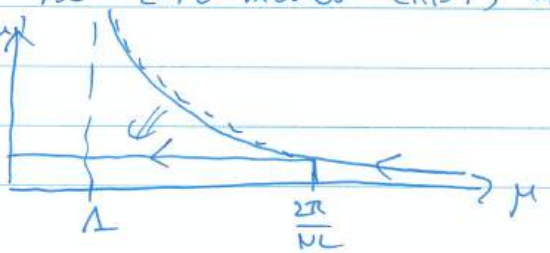
$$\Rightarrow \begin{cases} a_I^{(p_3,p_4)}(x, x_3 + L_3, x_4) = \omega^{p_3}\, a_I^{(p_3,p_4)}(x, x_3, x_4) \\ a_I^{(p_3,p_4)}(x, x_3, x_4 + L_4) = \omega^{p_4}\, a_I^{(p_3,p_4)}(x, x_3, x_4). \end{cases}$$

Thus, the 2d KK mass should be

$$m_{KK}^2 = \left(\frac{2\pi(N n_3 + p_3)}{N L_3}\right)^2 + \left(\frac{2\pi(N n_4 + p_4)}{N L_4}\right)^2.$$

As $(p_3, p_4) \neq (0,0)$, no zero modes exist, w/ the smallest gap $\frac{2\pi}{NL}$.

$\Rightarrow$ Genuinely weak-coupling setup!



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To achieve the adiabatic continuity, we still have to show
- ① Area law for the 2d Wilson loop
- ② Multi-branch θ-vacua

These properties turn out also to be true, but cannot be explained within the classical & perturbative analysis. [Within classical analysis, W_{2d} obeys perimeter law & $(\mathbb{Z}_N^{(1)})_{2d}$ is spontaneously broken]
Non-perturbative analysis with fractional instantons plays the crucial role, and we are going to discuss the details of the fractional instanton.

Before going into the semiclassical analysis, I was confident that this should work without computations.
Symmetry was the key for the confidence.
4d YM anomaly: $Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B} Z_\theta[B]$.

Under T^2-compactification, B splits into the following ingredient

$$B_{4d} = \underbrace{B_{2d}}_{(\mathbb{Z}_N^{(1)})_{2d}} + \underbrace{A^{(3)}\wedge\frac{dx_3}{L_3} + A^{(4)}\wedge\frac{dx_4}{L_4}}_{(\mathbb{Z}_N\times\mathbb{Z}_N)^{(0)}_{2d}} + \underbrace{\frac{2\pi p}{N}}_{\text{'t Hooft flux}}\frac{dx_3dx_4}{L_3L_4}.$$

Substituting this expression into the 4d anomaly relation, we have

$$Z^{(T^2)_{p\text{-flux}}}_{\theta+2\pi}[B_{2d}, A^{(3)}, A^{(4)}] = e^{i\left[p\int B_{2d} - \frac{N}{2\pi}\int A^{(3)}\wedge A^{(4)}\right]} Z^{(T^2)_{p\text{-flux}}}_{\theta}[B_{2d}, A^{(3)}, A^{(4)}].$$

⇝ Only if we choose the 't Hooft flux p w/ $\gcd(N,p)=1$, the anomaly between the 2d 1-form symmetry & θ-periodicity survives. (cf. YT, Misumi, Sakai '17, Yamazaki '17).

Thus, the activation of the non-trivial 't Hooft flux ensures the multi-branch θ-vacua if we can show the area law of $W_{2d}(C)$.

Moreover, SSB of 2d 1-form sym is as difficult as SSB in QM.
In the case of pure YM, no anomaly requires SSB of 1-form sym, so the 1-form sym. should be restored.

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(iii) Fractional instanton.

BPS eq. for 4d YM:

$$\begin{aligned} S_{YM} &= \frac{1}{g^2}\int \mathrm{tr}(F\wedge *F) \\ &= \frac{1}{g^2}\int \left\{\frac{1}{2}\mathrm{tr}\left[(F\mp *F)\wedge *(F\mp *F)\right] \pm \mathrm{tr}(F\wedge F)\right\} \\ &= \frac{1}{2g^2}\int \mathrm{tr}\left[(F\mp *F)\wedge *(F\mp *F)\right] \pm \frac{8\pi^2}{g^2} Q_{top}. \end{aligned}$$

Thus.

$$\begin{cases} Q_{top} > 0 \Rightarrow S_{YM} \geq \frac{8\pi}{g^2} Q_{top} \ \& = \text{holds iff } F = *F \\ Q_{top} < 0 \Rightarrow S_{YM} \geq \frac{8\pi}{g^2}(-Q_{top}) \ \& = \text{holds iff } F = -*F. \end{cases}$$

Here, I will show (or, more precisely, argue) that there exists the self-dual solution (i.e. $F = *F$) on $\mathbb{R}^2 \times (T^1)_{\text{twisted}}$ satisfying

① $\mathbb{R}^2$ with $\sim NL$, (i) — looks like vortex in 2d

② $Q_{top} = \frac{1}{N}$, $S_{YM} = \frac{8\pi^2}{Ng^2}$ — fractional topological charge

③ (ii) W $\Rightarrow$ (iii) $e^{\frac{2\pi i}{N}} \times W$ — center vortex.

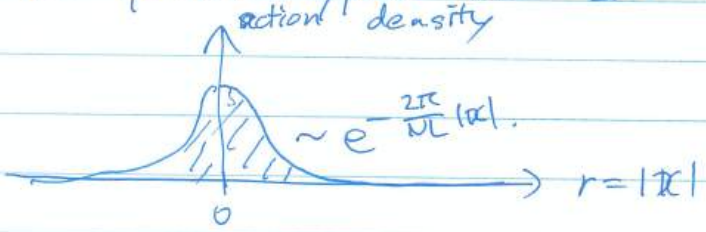
[* No one has succeeded to give an analytical expression for $F = *F$ on $\mathbb{R}^2 \times T^2_{\text{twisted}}$.
Thus, what I'm going to show here is the "best" convincing evidence for the self-dual center-vortex solution.]

(1) Vortex with the specific size $\frac{NL}{2\pi}$

The perturbative spectrum has the mass gap $\frac{2\pi}{NL}$.

$\Rightarrow$ Any accumulation of the action density should decay to the vacuum configuration as $\sim \exp(-\frac{2\pi}{NL}|x|)$.

$\therefore$ If any stable soliton exists, it should look like the vortex with the specific typical size $\sim \frac{NL}{2\pi}$.



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② Center vortex as a topological soliton.

In the $\mathbb{R}^2 \times (T^2)_{\text{twisted}}$ setup, $(\mathbb{Z}_N^{(1)})_{2d}$ is spontaneously broken at the classical level due to adjoint Higgsing:

To see this, it is convenient to consider $\mathbb{R} \times S^1 \times (T^2)_{\text{twisted}}$ w/ $L_2 \gg L = L_{3,4}$.

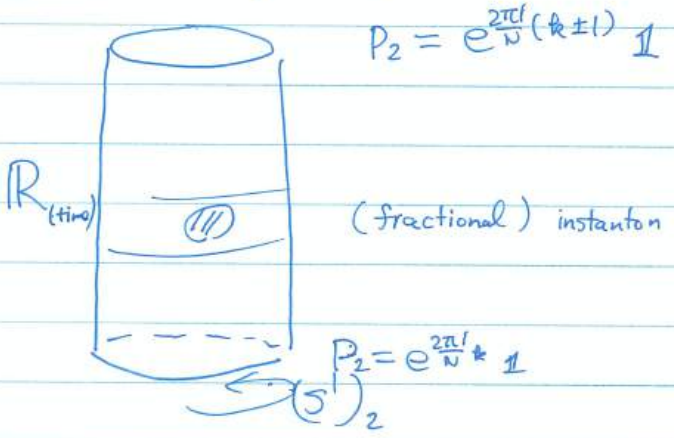
$$P_3 = S, \quad P_4 = C.$$

Flatness condition $F_{23} = 0$, $F_{24} = 0$ requires

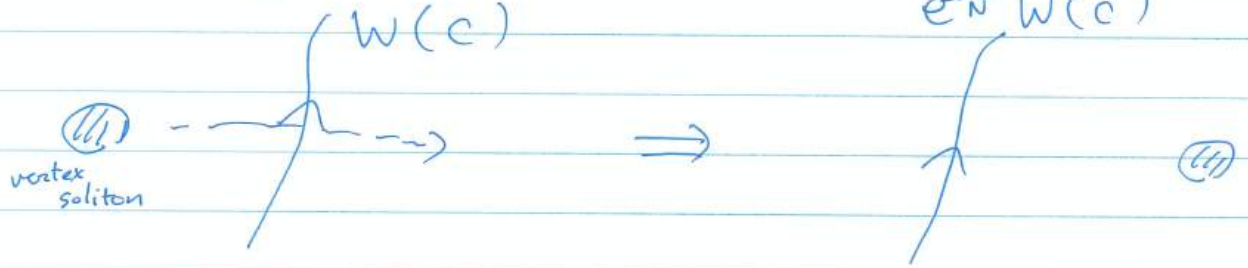
$$P_2 P_3 = P_3 P_2 \quad \& \quad P_2 P_4 = P_4 P_2.$$

$$\Rightarrow P_2 = e^{\frac{2\pi i}{N} k} \mathbb{1} \quad (k = 1, \cdots, N).$$

Thus, there should exist a topological soliton connecting these symmetry-broken vacua



$\Rightarrow$ This topological soliton have to be a center vortex:



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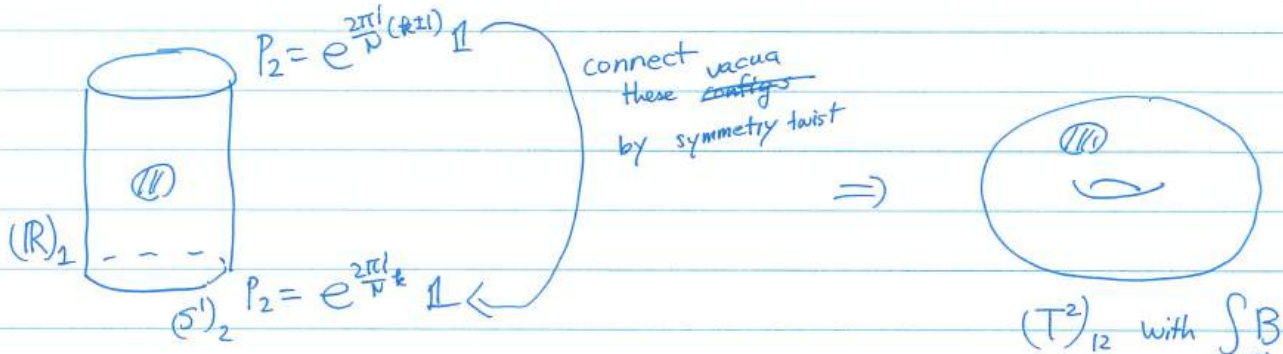
③ Fractional topological charge $Q_{top} = \pm 1/N$ & self-duality.



$$Q_{top}[B] = -\frac{N}{8\pi^2}\int_{T^4} B\wedge B + \mathbb{Z}$$

$$-\frac{N}{(2\pi)^2}\int_{(T^2)_{12}} B \int_{(T^2)_{34}} B$$

($\int_{T^4} B\wedge B = 1$, $\int_{(T^2)_{12}} B = \pm\frac{2\pi}{N}$, $\int_{(T^2)_{34}} B = \frac{2\pi}{N}$)

$$= \mp\frac{1}{N} + \mathbb{Z}.$$

Thus, the possible minimal topological charge is $Q_{top} = \mp\frac{1}{N}$.

$\Rightarrow$ By the BPS inequality,

$$S_{vortex} \geq \frac{8\pi^2}{g^2}|Q_{top}| = \frac{8\pi^2}{Ng^2}.$$

All these arguments ①–③ are rigorous (at least for physicists).
The remaining problem is:
Is the BPS bound saturated?

In the following, we will answer this positively by constructing the center vortex from the Kraan-van Baal-Lee-Lu-Yi (KvBLLY) monopole-instanton on $\mathbb{R}^3\times S^1$.
[Hayashi, YT '24]

Numerical lattice construction for the self-dual fractional instanton has been done in many papers (cf. Garcia Perez, Gonzalez-Arroyo, Soderberg '90, ... on $\mathbb{R}\times T^3$; Gonzalez-Arroyo, Montero '98, ... on $\mathbb{R}^2\times T^2$).
Taking the classical continuum limit, those solutions satisfy the BPS bound (within reasonable numerical accuracy).

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- Construction of the self-dual center vortex on $\mathbb{R}^2 \times T^2$ out of KvBLLY monopoles

Let us consider the geometry $\mathbb{R}^2 \times \underbrace{(S^1)_3 \times (S^1)_4}_{\text{'t Hooft twist } \int_{T^2} B = \frac{2\pi}{N}}$ w/ $L_3 \gg L_4$.

With this hierarchy of scales, it is natural to consider this compactificatn in two steps

$$\mathbb{R}^4 \Rightarrow \mathbb{R}^3 \times (S^1)_4 \Rightarrow \mathbb{R}^2 \times \overbrace{(S^1)_3 \times (S^1)_4}^{\text{'t Hooft twist}}.$$

(w/ $P_4|_\infty = C$)

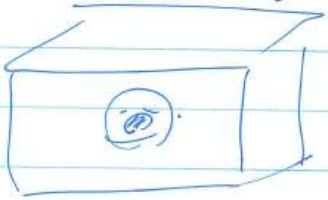
Self-dual YM solutions on $\mathbb{R}^3 \times S^1$ w/ non-trivial holonomy background is well-known thanks to (K.-M. Lee, P. Yi '97; K.-M. Lee, C.-h. Lu '98; Kraan, van Baal '98) $\Rightarrow$ KvBLLY calorons

$$P_4 = \omega^{-\frac{N-1}{2}} \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} = \mathrm{diag}\left(e^{-i\frac{2\pi}{N}\left(\frac{N-1}{2}\right)}, e^{-i\frac{2\pi}{N}\left(\frac{N-1}{2}\right)+\frac{2\pi i}{N}}, \cdots, e^{i\frac{2\pi}{N}\frac{N-1}{2}} \right)$$

$\rightsquigarrow$ off-diagonal gluons $a_I^{i\neq j}$ acquire the KK mass $\frac{2\pi(i-j)}{NL}$, and thus we have

3d $U(1)^{N-1}$ gauge theory described by diagonal gluons $a_I^{ii} =: a_I^i$

$\Rightarrow$ We have 't Hooft-Polyakov monopoles as the BPS solution.



$$\int_{S^1} d\vec{a} = 2\pi \vec{\alpha}_i \quad (i=1,\cdots,N-1)$$

$\vec{\alpha}_i$: positive simple root.

$(0,\cdots,0,\underset{i}{1},\underset{i+1}{-1},0,\cdots,0)$

In addition, KvBLLY noticed that there exists one extra fundamental monopole!

To see this, we should understand how the 0-form center symmetry acts on 3d $U(1)^{N-1}$ gauge fields.

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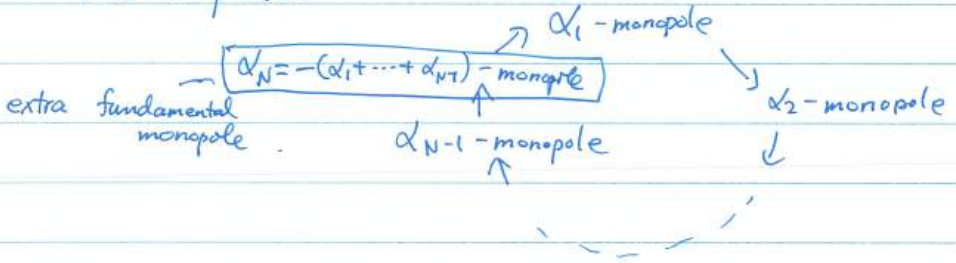
Naively, $\mathbb{Z}_N^{(0)}$ center sym. acts as $P_4 \mapsto \omega P_4$.
However, we have derived the 3d $U(1)^{N-1}$ gauge theory as an effective description after choosing the gauge $P_4 = \omega^{\frac{N-1}{2}} C$.

To find the correct action of $\mathbb{Z}_N^{(0)}$ for the 3d description, we should combine the naive action with a gauge transformation so that P_4 is unchanged:

$$P_4 \mapsto \omega P_4 \longmapsto \omega \underbrace{S C S^{-1}}_{P_4} = \omega \omega^{\frac{N-1}{2}} C S^{-1} S = P_4.$$

$\rightsquigarrow$ Under $\mathbb{Z}_N^{(0)}$ center sym., $a_I^i \mapsto a_I^{i+1}$.

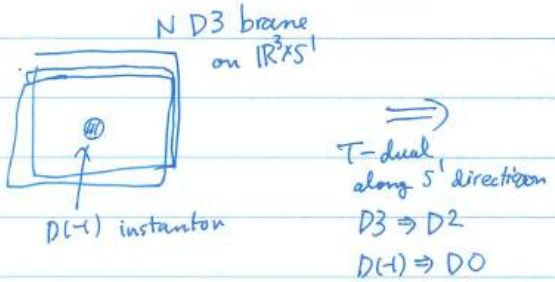
This $\mathbb{Z}_N^{(0)}$ symmetry then should cyclically permute monopoles

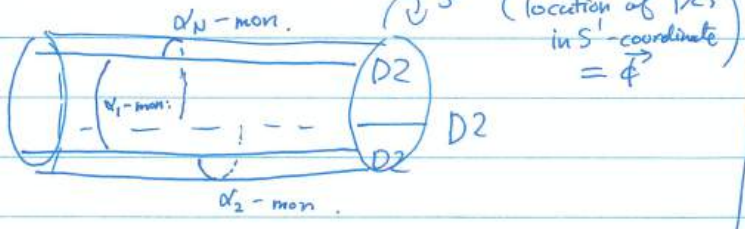


All of them (w/ $P_4 = \omega^{-\frac{N-1}{2}} C$ background) has

$$\begin{cases} Q_{top} = \frac{1}{N} \\ S_{YM} = \frac{8\pi^2}{g^2 N} \end{cases}.$$

In original papers, this fact is found via the D-brane interpretation of the ADHM/Nahm construction.





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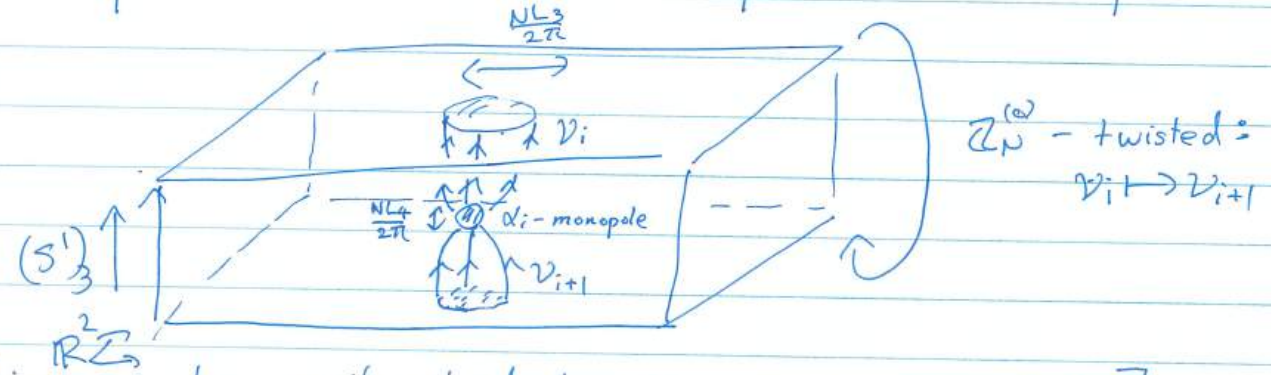
Now, we can perform the 2nd compactification step

$$\mathbb{R}^3 \times (S^1)_4 \Rightarrow \mathbb{R}^2 \times \overbrace{(S^1)_3}^{\text{'t Hooft twist.}} \times (S^1)_4$$

In the 3d perspective,
Insertion of the 't Hooft twist along $(S^1)_3 \times (S^1)_4$
$= \mathbb{Z}_N^{(0)}$ - twisted b.c. for $(S^1)_3$.

Let's put $\alpha_i (= \nu_i - \nu_{i+1})$ - monopole in this compactified setup.



[This can be easily solved by the mirror-image method.]

As long as $L_3 \gg L_4$, this construction shows

- $Q_{top} = \frac{1}{N}$
- $S_{YM} = \frac{8\pi^2}{g^2 N}$
- $W(C) = e^{\pm \frac{2\pi i}{N}}$.

By the index theorem, there exist 4 moduli for this solution

- 2 : translational moduli
- 2 : compact internal moduli

Unlike 4d instanton, it does not have the size moduli, which allows us to perform the dilute gas approximation in a reliable manner.

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⑩ Dilute center-vortex gas approximation, multi-branch θ-vacua & area law of the 2d Wilson loop.

We are now ready to perform the dilute gas approximation for the partition function using the center-vortex fractional instanton. (Basically, we repeat the same calc. for QM on S^1 w/ θ-term).

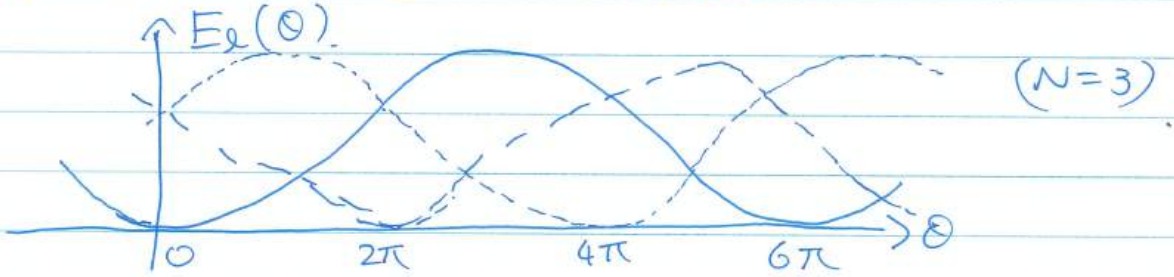
Let's compactify $\mathbb{R}^2 \times (T^2)_{\text{twisted}} \Rightarrow M_2 \times (T^2)_{\text{twisted}}$ w/ large M_2 to avoid IR divergences.

We set V = volume (M_2), then

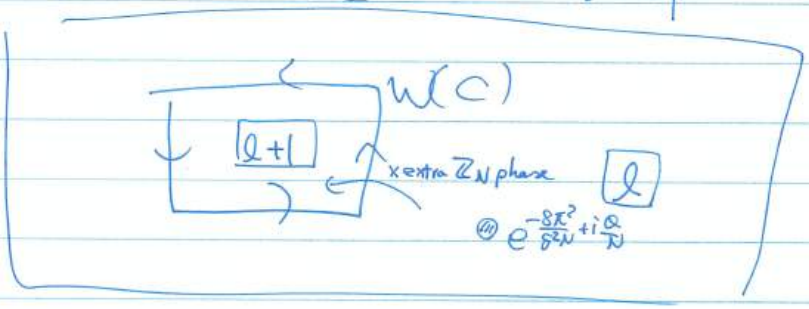
$$Z = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \left(Vke^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}\right)^n \left(Vke^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}\right)$$

$$= \sum_{\ell=0}^{N-1} \sum_{n,\bar{n}\geq 0} \frac{1}{n!\,\bar{n}!} e^{\frac{2\pi i}{N}\ell(n-\bar{n})} \left(Vke^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}\right)^n \left(Vke^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}\right)$$

$$= \sum_{\ell=0}^{N-1} \exp\left[-V\left(-2ke^{-\frac{8\pi^2}{g^2N}}\cos\left(\frac{\theta+2\pi\ell}{N}\right)\right)\right].$$



If we also have the Wilson loop



Wilson loop has the string tension $\sigma = E_{\ell+1}(\theta) - E_\ell(\theta)$, when $E_\ell(\theta)$ is the lowest-energy state.

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We've seen that YM partition func. contains the sum over the $\mathbb{Z}_N$ label ℓ, which comes out of $Q_{top} \in \mathbb{Z} \rightsquigarrow \delta_{n-\bar{n} \in N\mathbb{Z}} = \sum_{\ell=0}^{N-1} e^{\frac{2\pi i}{N}(n-\bar{n})\ell}$.

To understand its 4d meaning, let us introduce B_{2d} for M_2-direction ($\int_{M_2} B_{2d} = m$).

$\Rightarrow Q_{top} = -\frac{m}{N} + \mathbb{Z}$, and the Kronecker's delta should be modified as

$$\delta_{n-\bar{n} \in -m+N\mathbb{Z}} = \sum_{\ell=0}^{N-1} e^{\frac{2\pi i}{N}(n-\bar{n}+m)\ell}.$$

$$\therefore Z_\theta[B_{2d}] = \sum_{\ell=0}^{N-1} \underbrace{e^{-i\ell\int B_{2d}}}_{\text{2d SPT action}} \cdot \exp\left[-V\left(-2K e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\frac{\theta+2\pi\ell}{N}\right)\right)\right].$$

This 2d SPT action has the natural 4d origin

$$e^{-i\frac{N\ell}{4\pi}\int B\wedge B} \overset{M_2 \times (T^2)_{\text{twisted}}}{\Longrightarrow} \exp\left(-i\ell\int_{M_2} B_{2d}\right).$$

Thus, the label ℓ specifies the level for the 4d SPT state w/ the $\mathbb{Z}_N^{(1)}$ symmetry, which corresponds to the $(e,m)=(-\ell,1)$ dyon-condensation vacuum. [For Wilson-'t Hooft classification in modern viewpoint see Nguyen, YT, Unsal '23]

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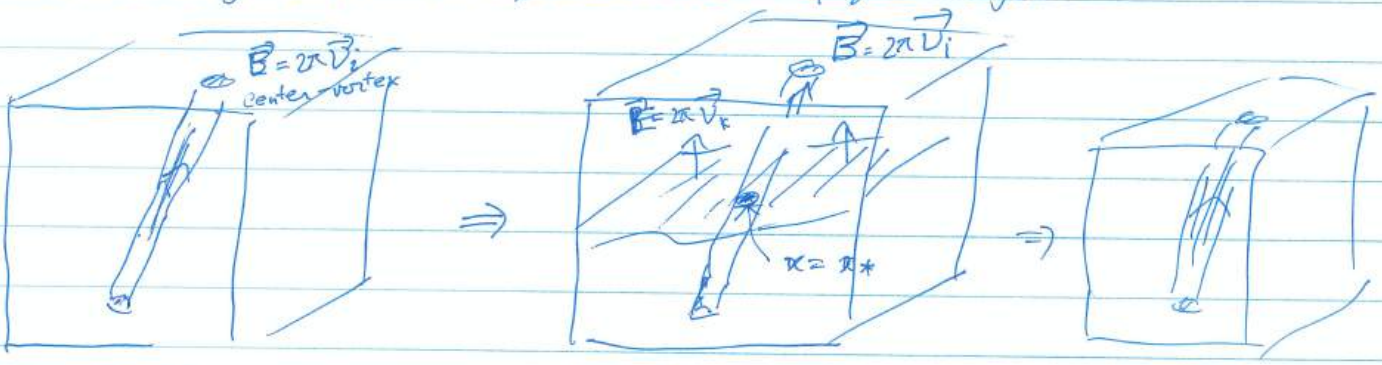
Addendum 4d interpretation of $\frac{1}{N}$-quantized topological charge.

So far, we've demonstrated the presence of the self-dual center vortex w/ $Q_{top} = \frac{1}{N}$ on $\mathbb{R}^2 \times (T^2)$ twisted.
Does such object exists in the 4d decompactified setup?

"If" we take the center-vortex proliferation scenario for confinement (cf. review by Greensite '03), there actually exists such $\frac{1}{N}$-fractionalized topological objects.



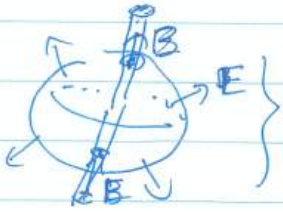
Euc. time $\tau_0 < \tau_*$ $\quad$ $\tau = \tau_*$ $\quad$ $\tau > \tau_*$

As in the above figure, let's consider a center vortex, whose worldsheet is extended along the time direction, and we then consider a process that an instantaneous center-vortex spatial worldsheet appears and disappears at $\tau = \tau_*$.

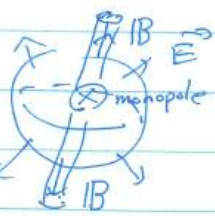
$$\Rightarrow \quad \frac{1}{(4\pi)^2} \vec{E} \cdot \vec{B} = \underbrace{\vec{\nu}_i \cdot \vec{\nu}_k}_{-\frac{1}{N} + \delta_{jk}} \delta(x - x_*)\, \delta(\tau - \tau_*)$$

$\therefore$ $\frac{1}{N}$-fractional topological charge appears at the intersection of two center-vortex worldsheets!

In this picture, the monopole worldlines ~~make~~ flip the orientation of the center-vortex worldsheets, which play the pivotal role to create the net topological charge. (Engelhardt, Reinhardt '00, Cornwall '00)



total $E \cdot B$ cancels



$\int \vec{E} \cdot \vec{B} = 1$

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Extension of the $\mathbb{R}^2 \times (T^2)_{\text{twisted}}$ semiclassics for QCD

We now consider to apply the center-vortex semiclassics on $\mathbb{R}^2 \times (T^2)_{\text{twisted}}$ for QCD. ⇝ How can we activate its 't Hooft flux without 1-form symmetry?
To this end, we first need to understand the global symmetry of QCD correctly including its periodicity.

QCD Lagrangian (in the chiral limit)

$$S = \frac{1}{g^2}\mathrm{tr}(F \wedge *F) + i\frac{\theta}{8\pi^2}\int \mathrm{tr}(F\wedge F) + \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu D_\mu \psi_i$$

$(D_\mu = \partial_\mu + i a_\mu)$.

Classical symmetry of this Lagrangian is

$$\frac{U(N_f)_L \times U(N_f)_R}{\mathbb{Z}_{N_c}}.$$

Since $\psi_i \mapsto e^{\frac{2\pi i}{N_c}} \psi_i$ can be generated both by $U(N_f)_V$ and $SU(N_c)_{\text{gauge}}$, we need to take the $\mathbb{Z}_{N_c}$ quotient.
⇔ All the gauge-invariant local operators has the quark number $\in N_c\mathbb{Z}$.

Quantum mechanically, $U(1)_A$ is explicitly violated to $(\mathbb{Z}_{2N_f})_A$, which is completely contained in the other continuous symmetries.
Thus

$$G_{QCD} = \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V}{\mathbb{Z}_{N_f} \times \mathbb{Z}_{N_c}}$$

In particular,

$$U(1)_{\text{baryon}} = \frac{U(1)_V}{\mathbb{Z}_{N_c}}.$$

In the language of the background gauge field, this quotient implies

$$\underbrace{dA_B}_{\substack{\text{canonically quantized}\\ \text{baryon-number } U(1) \text{ gauge field}}} = N_c \Big(\underbrace{dA_V}_{\substack{\text{canonically quantized}\\ \text{quark-number } U(1) \text{ gauge field}}} + \underbrace{B_c}_{\text{color 't Hooft flux}} \Big) \qquad [\text{YT, '18}]$$

⇒ Even though QCD has no 1-form symmetry, we can activate the color 't Hooft flux by setting $\int_{2\text{-cycle}} dA_B = 2\pi$.

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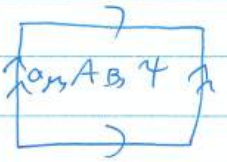
Since this property is crucial for semiclassics of QCD on $\mathbb{R}^2 \times T^2$, let us confirm it in an elementary way.

Let's put the fundamental quark w/ $Q_B = \frac{1}{N}$ on T^2.



We introduce the color & $U(1)_B$ transition functions

$$\begin{cases} a_\mu(x_3+L_3, x_4) = g_3^{-1}(x_4)\, a_\mu(x_3,x_4)\, g_3(x_4) - i g_3^{-1}(x_4)\,\partial_\mu g_3(x_4) \\ a_\mu(x_3, x_4+L_4) = g_4^{-1}(x_3)\, a_\mu(x_3,x_4)\, g_4(x_3) - i g_4^{-1}(x_3)\,\partial_\mu g_4(x_3) \end{cases}$$

$$\begin{cases} A_{B\mu}(x_3+L_3, x_4) = A_{B\mu}(x_3,x_4) + \partial_\mu \phi_3(x_4) \\ A_{B\mu}(x_3, x_4+L_4) = A_{B\mu}(x_3,x_4) + \partial_\mu \phi_4(x_3) \end{cases}$$

For the quark field

$$\begin{cases} \psi(x_3+L_3, x_4) = g_3^{-1}(x_4)\, e^{-\frac{i}{N}\phi_3(x_4)}\, \psi(x_3,x_4) \\ \psi(x_3, x_4+L_4) = g_4^{-1}(x_3)\, e^{-\frac{i}{N}\phi_4(x_3)}\, \psi(x_3,x_4) \end{cases}$$

$\Rightarrow$ Cocycle condition for the quark field:

$$\begin{aligned} \psi(L_3,L_4) &= g_3^{-1}(L_4)\, e^{-\frac{i}{N}\phi_3(L_4)}\, \psi(0,L_4) \\ &= g_3^{-1}(L_4)\, e^{-\frac{i}{N}\phi_3(L_4)}\, g_4^{-1}(0)\, e^{-\frac{i}{N}\phi_4(0)}\, \psi(0,0) \\ g_4^{-1}(L_3) e^{-\frac{i}{N}\phi_4(L_3)}\, \psi(L_3,0) &= g_4^{-1}(L_3)\, e^{-\frac{i}{N}\phi_4(L_3)}\, g_3^{-1}(0)\, e^{-\frac{i}{N}\phi_3(0)}\, \psi(0,0) \end{aligned}$$

$$\therefore\ g_3^{-1}(L_4)\, g_4^{-1}(0) = g_4^{-1}(L_3)\, g_3^{-1}(0) \cdot e^{\frac{i}{N}\left[(\phi_3(L_4)-\phi_3(0)) - (\phi_4(L_3)-\phi_4(0))\right]}$$

$$= \frac{1}{2\pi}\int_{T^2} dA_B$$

Thus, if $\frac{1}{2\pi}\int dA_B = P \neq 0 \mod N$, we find

$$g_3^{-1}(L_4)\, g_4^{-1}(0) = g_4^{-1}(L_3)\, g_3^{-1}(0)\, e^{\frac{2\pi i}{N}P}$$

$e^{\frac{2\pi i}{N}P}$ $\leftarrow$ 't Hooft flux!!

Similar thing occurs for the non-Abelian flavor symmetry $SU(N_f)_V$ when $\gcd(N_c, N_f)$ is nontrivial (especially when $N_c = N_f$).
For this case, taking the shift & clock (flavor-twisted) b.c. for T^2 requires us introduce the shift & clock (color-twisted) b.c.
In this lecture, we focus on the $U(1)_B$ case.

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(III) QCD on $\mathbb{R}^2 \times (T^2)_{U(1)_B\text{-flux}}$ = $\mathbb{Z}_N$ - Schwinger model.

Due to $dA_B = N_c(dA_V + B_c)$, $\int_{T^2} dA_B = 2\pi$ gives $\int_{T^2} B_c = \frac{2\pi}{N_c}$.

⇒ Gauge sector is
- classically $\mathbb{Z}_N^{(0)}$-center symmetric vacuum $P_3 = S$, $P_4 = C$
- spectrum is perturbatively gapped by $\frac{2\pi}{NL_{3,4}}$.
- self-dual center vortex exists, and the dominant configuration is given by its ensemble.

We then would like to understand how quarks enter in this picture.
First, we should look at if there exists 2d zero-modes, which is equivalent to solve

$$\underbrace{\not{D}_{T^2}}_{\gamma_3 D_3 + \gamma_4 D_4} \psi = 0.$$

Index theorem tells

$$\text{Index}(\not{D}_{T^2}) = \frac{1}{2\pi}\int_{T^2} dA_B = 1,$$

so there exists one zero mode for the positive chirality in the sense of $i\gamma_3\gamma_4$ $\left(\begin{smallmatrix}\sigma_3 & \\ & -\sigma_3\end{smallmatrix}\right)$

⇒ Starting from N_c-color 4-component spinor (i.e. 4d Dirac spinor), we obtain a single 2-component spinor (i.e. 2d Dirac spinor).

This shows that
QCD on $\mathbb{R}^2 \times (T^2)_{U(1)_B\text{-flux}}$
= 2d Dirac fermion w/ the gas of $\mathbb{Z}_N$ vortices
$\cong$ 2d U(1) gauge field w/ charge-N Higgs field
$\cong$ 2d Schwinger model w/ charge-N Higgs field.

© Historically, Schwinger model played an important role as a toy model of 4d QCD. It actually describes the "QCD" in a special circumstance! ©

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If we set the uniform $U(1)_B$-flux background

$$dA_B = \frac{2\pi}{L^2} dx_3 dx_4 ,$$

we can take a gauge

$$A_B = \frac{2\pi}{L^2} x_3 dx_4 .$$

Then, the Dirac zero-mode equation becomes

$$\left[\gamma_3 \partial_3 + \gamma_4 \left(\partial_4 + i \tfrac{1}{N} A_{B,4} \right) \right] \psi = 0$$

w/ b.c.

$$\psi(x_3 + L, x_4) = S^{-1} e^{-i \frac{2\pi i}{NL} x_4} \psi(x_3, x_4), \quad \psi(x_3, x_4 + L_4) = C^{-1} \psi(x_3, x_4)$$

By projecting it into the $i\gamma_3\gamma_4 = \pm 1$ sectors, the zero-mode eq. becomes

$$\left[\partial_3 \mp i \left(\partial_4 + i \frac{2\pi}{NL^2} x_3 \right) \right] \psi_\pm(x_3, x_4) = 0 .$$

It is then the standard exercise to confirm ψ_+ has a solution while ψ_- does not.
This confirms the result of the index thm explicitly.

(III) Derivation of chiral Lagrangian.

Let us perform the semiclassical analysis using the dilute center-vortex gas approximation.

For simplicity, we begin with the 1-flavor case.
Later we extend to the N_f-flavor case (which uses non-Abelian bosonizati).

In the above, we've seen that

$$\text{QCD on } \mathbb{R}^2 \times (T^2)_{U(1)_B\text{-flux}} = \begin{cases} \text{gauge sector : center-vortex gas} \\ \text{quark sector : a single 2d Dirac fermion.} \end{cases}$$

and we must study the interaction between quarks & center-vortex.

The convenient trick is the 2d Abelian bosonization

$$\bar\psi \gamma^\mu \partial_\mu \psi \quad \longleftrightarrow \quad \frac{1}{8\pi} (\partial_\mu \eta')^2$$

$$\bar\psi_L \psi_R \quad \longleftrightarrow \quad e^{i\eta'}$$

$$U(1)_A : \psi \mapsto e^{i\alpha\gamma_5}\psi, \ \bar\psi \mapsto \bar\psi e^{i\alpha\gamma_5} \quad \longleftrightarrow \quad U(1)_{\text{shift}} : \eta' \mapsto \eta' + 2\alpha .$$

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QCD has the spurious symmetry

$$\psi \mapsto e^{i\alpha\gamma_5}\psi, \quad \bar{\psi} \mapsto \bar{\psi} e^{i\alpha\gamma_5}, \quad \theta \mapsto \theta - 2\alpha.$$

This has to be respected also by the low-energy Lagrangian.

∴ Center-vortex vertex $\sim e^{-\frac{8\pi^2}{Ng^2}} e^{i\frac{\eta'+\theta}{N}}$.

$$Z_{QCD} = \sum_{l=0}^{N-1} \sum_{n,\bar{n}\geq 0} \frac{e^{\frac{2\pi i l}{N}(n-\bar{n})}}{n!\,\bar{n}!} \times \int \mathcal{D}\eta' e^{-\int \frac{1}{8\pi}(\partial_\mu \eta')^2} \left(\int d^2x\, k e^{-\frac{8\pi}{Ng^2}} e^{i\frac{\eta'+\theta}{N}}\right)^n \times \left(\int d^2x\, k\, e^{-\frac{8\pi}{Ng^2}} e^{-i\frac{\eta'+\theta}{N}}\right)^{\bar{n}}$$

$$= \sum_{l=0}^{N-1} \int \mathcal{D}\eta' \exp\left[-\int d^2x \left\{\frac{1}{8\pi}(\partial_\mu\eta')^2 - 2k e^{-\frac{8\pi}{Ng^2}} \cos\left(\frac{\eta'+\theta+2\pi l}{N}\right)\right\}\right]$$

[YT, Ünsal '22; Hayashi, YT '24]

(i) Center-vortex gives the mass term for η' (plase of the chiral condensate)
⇒ Resdution of the $U(1)_A$ problem.

(ii) The potential $V(\eta', l) = -2k e^{-\frac{8\pi}{Ng^2}} \cos\left(\frac{\eta'+\theta+2\pi l}{N}\right)$

contains not only η' but also the $\mathbb{Z}_N$ label l.
Moreover, the periodicity of η' becomes nontrivial:

$$\begin{cases} (\eta'+2\pi,\ l) \sim (\eta',\ l+1) \\ (\eta',\ l+N) \sim (\eta',\ l). \end{cases}$$

Its consequence will be dicussed in more details later.
[← The same thing occurs for 2d Schwinger model. See, e.g., Appendix of Hayashi, YT '24.]

(iii) Conventional Kobayashi-Maskawa-'t Hooft vertex by instanton takes the form of

$$V_{\text{instanton}}(\eta') \approx -e^{-\frac{8\pi}{g^2}} \cos(\eta'+\theta).$$

Our result suggets its $\frac{1}{N}$ fractionalization is the correct one.

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The extension to N_f-flavor is straightforward if we use the non-Abelian bosonization.

N_f Dirac fermions $\sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i$ $\quad\Leftrightarrow\quad$ $U(N_f)_1$ WZW CFT.

$$\frac{1}{8\pi}\int_{M_2} \mathrm{tr}\left(\partial_\mu \tilde{U}^{-1}\partial_\mu \tilde{U}\right) + \frac{i}{12\pi}\int_{M_3}\mathrm{tr}\left[(\tilde{U}^\dagger d\tilde{U})^3\right]$$

w/ $\partial M_3 = M_2$.

$\left(\tilde{U} = e^{i\eta'/N_f}\cdot \overset{SU(N_f)}{U}\right)$.

Spurious symmetry tells the center-vortex vertex should look like

$$e^{-\frac{8\pi}{Ng^2}} e^{i\frac{\theta}{N}} \underbrace{(\det U)}_{e^{i\eta'}}{}^{1/N}.$$

$$\Rightarrow \quad V(\eta') \sim -e^{-\frac{8\pi}{Ng^2}} \cos\left(\frac{\eta' + \theta + 2\pi\ell}{N}\right).$$

Since η' becomes massive, we obtain $SU(N_f)_1$ WZW theory.

$\Leftarrow$ This can be also obtained from the 4d chiral Lagrangian, which should be valid if the size of $(T^2)_{U(1)_B\text{-flux}}$ is much larger than Λ^{-1}!

If N_f is not so large, we empirically know

$$SU(N_f)_L \times SU(N_f)_R \xrightarrow{SSB} SU(N_f)_V.$$

The chiral Lagrangian w/ WZW term:

$$\mathcal{L} = \frac{f_\pi^2}{2}\mathrm{tr}(\partial_\mu U^{-1}\partial_\mu U) + iA_{B\mu}J_\mu + \overset{5d}{WZW},$$

where

$$*J = \frac{1}{24\pi}\mathrm{tr}((U^\dagger dU)^3)$$

is the Skyrmion current.

For $\overset{M_2}{\mathbb{R}^2}\times(T^2)_{U(1)_B\text{-flux}}$,

$$\int_{M_2\times(T^2)} A_{B\mu}J_\mu = \int_{M_3} J \wedge \overbrace{\int_{T^2} dA_B}^{2\pi i} = i\frac{1}{12\pi}\int_{M_3}\mathrm{tr}((U^\dagger dU)^3).$$

and we obtain the $SU(N_f)_1$ WZW CFT.

Direct QCD calc. for small $\mathbb{R}^2\times(T^2)_{\text{flux}}$ = Chiral Lagrangian for large $\mathbb{R}^2\times(T^2)_{\text{flux}}$.

$\Leftarrow$ Adiabatic Continuity!

[YT, Ünsal '22]

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(m) η' – vortex, Fractional quantum-Hall droplet.

Let us come back to the 1-flavor QCD for clarity.

$$V(\eta') \sim -\cos\left(\frac{\eta' + \theta + 2\pi l}{N}\right).$$

As we've emphasized, this potential is not 2π-periodic for η', while $\bar{\psi}_L \psi_R = e^{i\eta'}$ says $\eta' \sim \eta' + 2\pi$ is a gauge redundancy.

In particular, if we consider the η' – vortex

the associated η'-kink requires the jump of the discrete label $l \mapsto l-1$ for consistency.

$\Big\|$

Wilson loop!

$\Rightarrow$ η'-vortex = end-point of the Wilson loop
= fundamental quark ψ.

This is the clear distinction when we compare our semiclassics and the conventional instanton vertex.

Here, all the story is given in the 2d effective theory, and it is natural to ask what is the 4d counterpart. That is Komargodski's quantum Hall droplet scenario.
[Komargodski '18].

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• Quantum Hall Droplets

For $N_f \geq 2$, it is well known that

Skyrmions = Baryons,

so baryons are described as topological solitons of Nambu-Goldstone bosons.

For $N_f = 1$, however, there is no counterpart.
This is partly obvious, since $U(1)_A$ is anomalously broken and no NG boson exists.
Still, one may take $N_c \to \infty$, where $U(1)_A$ violation is $\frac{1}{N_c}$-suppressed, and η' is the approximate NG boson.
But, no Skyrmion exists as $\pi_3(U(1)_A) = 0$.

Komargodski ('18) proposed the following set up:

Idea: $\eta' \sim \theta$.

In pure YM, θ and $\theta + 2\pi$ are distinct SPT, and the interface connecting them supports $U(1)_{-N}$ CS theory (= Quantum Hall Liquid!)

Due to the analogy between η' and θ, it would be natural to consider that the CS theory lives also on the η'-kink connecting $\eta' = \pm\pi$.

Then, by the gauge inv., the η'-vortex must support chiral fermion.
If we excite the boundary state by the vertex operator, the energy of this system is stabilized as

$$E \sim \underbrace{\frac{1}{L}}_{\text{bdry excitation}} + \underbrace{L^2}_{\text{tension of the surface}} + \underbrace{L}_{\text{tension of the vortex ring}}.$$

This stable state can be understood as the baryon for 1-flavor QCD.

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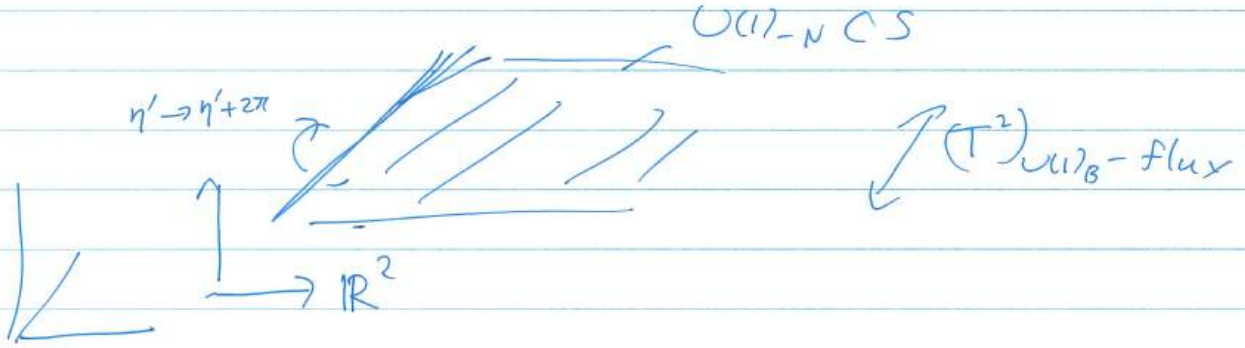
Let us then consider to put it on our $\mathbb{R}^2 \times (T^2)_{\text{flux}}$ setup as follows.

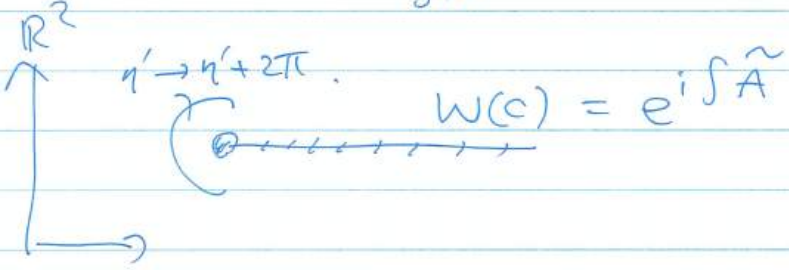


Since $dA_B = N_c (dA_V + B_c)$, $U(1)_N$ CS theory couples to dA_B as

$$\frac{N}{4\pi} \int \tilde{A}\, d\tilde{A} + \frac{1}{2\pi} \int \tilde{A} \wedge dA_B$$

$$\Downarrow \quad \text{int. out } (T^2)_{\text{flux - direction}}$$

$$\int \tilde{A}$$



This is exactly what we found for the η'-vortex in the 2d semiclassics !